The Cost of Simple Bidding in Combinatorial Auctions

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We study the complexity of bidding optimally in one-shot combinatorial auction mechanisms. Specifically, we consider the two most-commonly used payment rules: *first-price* and *VCG-nearest*. Prior work has largely assumed that bidders only submit bids on their bundles of interest. However, we show the surprising result that a single-minded bidder may lose an exponential amount of utility by playing his optimal *simple* strategy (only bidding on his bundle of interest) compared to playing his optimal *complex* strategy (which involves bidding on an exponential number of bundles). Our work suggests that it is important for future research on combinatorial auctions to fully take these effects into account.

1 INTRODUCTION

Imagine you are in a Turkish bazaar and you have very simple preferences: you want to buy a lamp, but you are not interested in any other item. Given the nature of the marketplace you are in, you naturally expect that you have to negotiate with the seller to obtain the best possible price. However, it would be absurd if you would also have to negotiate regarding all possible bundles of items in the entire bazaar to obtain the best price for the lamp! In this paper, we show that this phenomenon may arise in combinatorial auction mechanisms: a bidder who is only interested in a single item may have to submit a very "complex" bid (expressing values on an exponential number of bundles) to maximize his utility, and submitting a "simple" bid instead can lead to a loss of an exponential amount of utility.

Our work is related to a recent thread of research that aims to understand what makes mechanisms "simple to play", by focusing on obvious strategyproofness and related concepts (see, e.g., [Li, 2017], [Pycia and Troyan, 2019] and [Börgers and Li, 2019]). The notion of simplicity we develop is based on the information content of the actions taken by agents, which is indirectly linked to the uncertainty in the environment these agents find themselves in. To make this concrete, we focus specifically on combinatorial auctions, a class of mechanisms that are widely used in practical applications.

Combinatorial auctions (CAs) are used in settings where multiple heterogeneous goods are being auctioned off to multiple bidders [Cramton et al., 2006]. In contrast to simpler designs like *simultaneous ascending auctions*, a CA allows bidders to express arbitrarily complex preferences over *bundles* of items, including substitutes and complements preferences. CAs are frequently used in practice, for example for the sale of industrial procurement contracts [Sandholm, 2013] or for selling TV ad slots [Goetzendorff et al., 2015].

One of the key challenges in CA design is to find a suitable mechanism to provide bidders with good incentives. At first sight, the well-known VCG auction [Clarke, 1971, Groves, 1973, Vickrey, 1961] may look like a suitable mechanism in the CA domain given that it is efficient and strategyproof. Unfortunately, VCG can produce very low or even zero revenue in domains with complements [Ausubel and Milgrom, 2006]. Therefore, the CA designs used in practice are non-strategyproof mechanisms.

The *first-price package auction* has been used in many different domains for over 20 years (e.g., to auction off bus routes [Cantillon and Pesendorfer, 2007], to procure milk supplies in Chile [Epstein et al., 2004], and to auction off wireless spectrum (e.g., in Norway in 2013 and France in 2011) [Kokott et al., 2017]). More recently, second-price style designs for CAs like the *combinatorial clock auction (CCA)* [Ausubel et al., 2006] have gained traction. The CCA uses the Vickrey-nearest payment rule, which (while not strategyproof) is designed to charge bidders payments that are in the core but also as close to VCG as possible [Day and Cramton, 2012, Day and Milgrom, 2008, Day and Raghavan, 2007]. The CCA has been used for over 10 years in many different countries to conduct spectrum auctions, generating over $15 Billion in revenues. More recently, it has also been used to auction off licenses for offshore wind farms [Ausubel et al., 2011].

Of course, CAs are known to be "complex" (at least in some respects). Most importantly, finding the efficient allocation may require exponential communication between the auctioneer and the bidders in the worst case [Nisan and Segal, 2006]; however, only if bidders have fully general preferences. Not surprisingly, iterative mechanisms like the CCA present some opportunities for manipulation (e.g., [Janssen and Karamychev, 2016]). And given that VCG-nearest is not strategyproof, it is not surprising that, in Bayes-Nash equilibrium, there may be opportunities for bidders for *bid shading, free-riding* or even *overbidding* to optimize their utility.[Ausubel and Baranov, 2019, Beck and Ott, 2013].

But these results should not distract us from the fact that CAs were originally designed to make bidding in the auction simpler for the bidders! For example, Cramton [2013] writes: *"In contrast [to the SAA], the combinatorial clock auction has more complex rules, but the rules have been carefully constructed to make participation especially easy."* We are interested in analyzing to what degree CAs truly make participation "easier for bidders."

One aspect that has been largely overlooked in the literature so far are bidders' incentives for bidding on items they have zero value for.[1] The research community has commonly assumed that it is sufficient for bidders to submit bids on their bundles of interest to maximize their utility. For example, Milgrom [2007] writes: *"[...] these sealed-bid package designs require that bidders name prices for each package of interest to them."* Similarly, Cramton [2013] writes: *"For the most part, the bidder can focus simply on determining its true preferences for packages that it can realistically expect to win."* In fact, prior theoretical and algorithmic work analyzing Bayes-Nash equilibria of CA payment rules has often assumed (explicitly or implicitly) that bidders only bid on their bundles of interest [Ausubel and Baranov, 2019, Bosshard et al., 2020, Rabinovich et al., 2013], with the notable exception of Beck and Ott [2013]. In this paper, we show that assuming that bidders only bid on their bundles of interest is a very significant restriction of the game-theoretic analysis, because bidder's best responses may no longer be optimal when the full strategy space (of bidding on all bundles) is considered. In the worst case, this leads to radically different bidder behavior with utilities that can differ by an exponential amount.

We study this question formally for the two most commonly-used CA payment rules: first-price and VCG-nearest. We first present an example where bidders are incentivized to submit bids that are more complex than their underlying valuations. This effect persists in Bayesian Nash Equilibrium: it allows bidders to coordinate with each other even in the face of uncertainty. The submitted bids can be interpreted as containing multiple "conditional bids" placed on the same item, such that only the most beneficial of these bids is activated and ends up winning in the auction.

Our main result pushes further in this direction and shows that, for a general CA, the amount of utility gain from complex bidding manipulations can be very significant in the worst case. A bidder who is interested in a single item (and thus has the simplest type of valuation possible) may be incentivized to bid on exponentially many bundles of items. Failing to do this, he may only obtain an exponentially small fraction of his possible utility. As a consequence, even small bidders with a localized interest in only a few items cannot avoid having to reason about the entire strategy space of the CA. This showcases a fundamental tension in auction design: when a series

[1] In lab experiments, Scheffel et al. [2011] have observed bidders engaging in "extended bundling", i.e., bidding on bundles that include items of no value to them. However, they attribute all of these actions to mistakes by bidders, and not to utility-increasing strategies.

of simple auctions are joined together into one large combinatorial auction, this helps large bidders with complex valuations, but it can actually hurt small bidders in unexpected ways, by needlessly exposing them to the full complexity of the more general mechanism.

It is surprising that our result also holds for the VCG-nearest payment rule, given that it is a minimum revenue core-selecting rule and thus minimizes the joint incentives to misreport of all bidders [Day and Raghavan, 2007]. The construction used in our result shows that this joint minimization can still leave individual bidders with very bad incentives, by setting their payment very high (equal to first-price) for no good reason.

Importantly, bidders are *not* pushed into this behavior by a mechanism taking place over many rounds, that elicits only limited information, or is restricted in some other way. Rather, they are making rational usage of the large strategy space provided to them by a direct revelation mechanism, i.e., a sealed-bid combinatorial auction with no limits on the number of bundles reported.

Finally, to put our results into context, we show that having uncertainty about other bidders' bids is a key ingredient for complex bidding to be beneficial. In the absence of such uncertainty, a bidder can always find a bid that is at most as complex as his true valuation, which maximizes his utility under any reasonable payment rule.

2 PRELIMINARIES

2.1 Formal Model

A combinatorial auction (CA) is used to sell a set $M = \{1, 2, \ldots, m\}$ of goods to a set $N = \{1, 2, \ldots, n\}$ of bidders. Each bidder i's preferences over bundles are captured via the bidder's *valuation* v_i. Specifically, for each bundle of goods $K \subseteq M$, we let $v_i(K) \in \mathbb{R}_{\geq 0}$ denote bidder i's value for bundle K. We normalize these values such that $v_i(\emptyset) = 0$. We assume that valuations satisfy *free disposal*, i.e., for any pair of bundles K, K' it holds that: $K \subseteq K' \Rightarrow v_i(K) \leq v_i(K')$.

Each bidder submits a bid b_i to the auction, which is a (possibly non-truthful) declaration of his whole valuation.[2] In this paper, we require bids to be submitted using the XOR bidding language [Nisan, 2006]. Thus, each bid is a set containing zero or more *atomic bids* (K, r), where K is a bundle and $r \in \mathbb{R}_{\geq 0}$ the *reported value* for that bundle. Because of the free-disposal assumption, the reported value of any bundle for which no atomic bid was submitted is implicitly inferred to be the maximum reported value of any of its sub-bundles, or zero if there exist no such sub-bundles. With a slight abuse of notation, we let $b_i(K)$ denote the value r which bidder i has explicitly (or implicitly) reported for bundle K. The bid profile $b = (b_1, \ldots, b_n)$ is the vector of all bids from all bidders. The bid profile of every bidder except bidder i is denoted b_{-i}.

The CA has an allocation rule X, determining an allocation $x = X(b)$, where $x_i = X_i(b)$ denotes the bundle assigned to bidder i. We require the resulting allocation to be *feasible*, i.e., $\forall i, j \in N : x_i \cap x_j = \emptyset$, and we let $\mathcal{F}$ denote the set of all feasible allocations. Furthermore, in this paper, we only consider *efficient* allocations (i.e., allocations that maximize *reported social welfare*, which is the sum of bidders' reported values). The CA also has a payment rule p which is a function assigning a payment $p_i(b) \in \mathbb{R}_{\geq 0}$ to each bidder i. Together, the allocation x and the payment vector p are called the auction *outcome*.

We let $u_i(v_i, b)$ denote bidder i's utility for an auction outcome, given his own valuation v_i and bid profile b. We assume that the utility function is *quasi-linear* of the form $u_i(v_i, b) = v_i(X_i(b)) - p_i(b)$. We assume that the auction is *individually rational (IR)*, i.e., the bidder's payment is less than or equal to his reported value for the bundle he obtains: $p_i(b) \leq b_i(X_i(b))$.

[2]To distinguish the auctioneer from the bidders, we use "she/her" when referring to the auctioneer and "he/his" when referring to the bidders.

2.2 Payment Rules and the Core

In this paper, we analyze two CA payment rules: *first-price* and *VCG-nearest*.

Definition 2.1. Given allocation rule X and bid profile b, bidder i's *first-price* payment is:

$$p_i(b) := b_i(X_i(b)). \tag{1}$$

To define VCG-nearest payments, we must first introduce VCG payments and the core. For this, we next define the set of feasible allocations when only considering a set of items $K \subseteq M$ as $\mathcal{F}_K$. We now define the reported social welfare achieved by a subset $L \subseteq N$ of bidders when only using items $K \subseteq M$ as follows:

$$W(b, L, K) := \max_{x \in \mathcal{F}_K} \sum_{j \in L} b_j(x_j). \tag{2}$$

Using this notation, we can now define standard VCG payments.

Definition 2.2. Given allocation X and bid profile b, bidder i's *VCG payment* is:

$$\mathrm{VCG}_i(b) := W(b, N \setminus \{i\}, M) - W(b, N \setminus \{i\}, M \setminus X_i(b)). \tag{3}$$

In words, i's payment is the reported social welfare achievable by all bidders except i, minus the welfare they still achieve in the presence of i.

Note that we define bidder i's VCG payments based on the bid profile b, and not based on the valuation profile v. This allows us to compute what the VCG payments *would have been*, even if a payment rule other than VCG is used in the auction itself.

Finally, note that $\mathrm{VCG}_i(b)$ only depends on bidder i's bid b_i via i's allocated bundle $K = X_i(b)$. We can thus equivalently write the VCG payment as $\mathrm{VCG}_i(b_{-i}, K)$. This is convenient when we later interpret this quantity as i's winning threshold for bundle K, i.e., for bidder i, $VCG_i(b_{-i}, K)$ is the minimum bid he needs to submit to win bundle K.

The Core. Informally, a payment vector p is said to be *outside the core* if a coalition of bidders is willing to pay more for the items than what the mechanism currently charges the winners. To avoid such outcomes, Day and Milgrom (2008) introduced the idea of *core-selecting payment* rules that restrict payments to be in the core.

Definition 2.3. Given allocation rule X and bid profile b, a price vector p is in the *core* if it satisfies the individual rationality constraints and the following additional core constraints:

$$\forall L \subseteq N : \sum_{j \in N \setminus L} p_j(b) \geq W(b, L, M) - W(b, L, X_L(b)). \tag{4}$$

Among all payment vectors in the core, the *minimum revenue core (MRC)* is the set of payment vectors with smallest L_1 norm, i.e., which minimize the sum of the payments of all bidders. We are now ready to define the second payment rule we study in this paper.

Definition 2.4. Given an allocation rule X and a bid profile b, the *VCG-nearest payment rule* is defined to compute the vector of payments in the minimum revenue core that minimizes the L_2 distance to the VCG payment vector.

2.3 Simplicity of Bids and Valuations

In this work, we are primarily interested in how the simplicity/complexity of a bidder's strategy impacts his utility. We measure the *simplicity* of a bid as the number of atomic bids it consists of. If a bid consists of k or fewer atoms, we call it k*-simple*. For the special case of $k = 1$, we call it *simple*.

We say that a bid is *complex* if it consists of 2^m or less atomics bids. We will equivalently say that a complex bid is 2^m-simple or ∞-simple (to simplify notation).

We introduce analogous terminology for valuations. We say that the *simplicity of a valuation* is the smallest simplicity of any bid that can fully capture this valuation (i.e., it can explicitly or implicitly declare the true value for each bundle).

2.4 Expected Utility and Best Responses

We consider CAs in a Bayesian setting, where bidder i knows his own valuation v_i, but he only has probabilistic information (i.e., a prior) over each other bidder j's bid b_j, represented by the distribution B_j. Given this uncertainty, the goal of bidder i is to maximize his *expected utility* $\bar{u}_i$ defined as

$$\bar{u}_i(v_i, b_i) := \mathop{\mathbb{E}}_{b_{\text{-}i} \sim B_{\text{-}i}} \left[u_i(v_i, b_i, b_{\text{-}i}) \right]. \tag{5}$$

Let $\mathcal{B}^k$ denote the set of all k-simple bids. We call the highest possible expected utility that can be achieved with any bid in $\mathcal{B}^k$ is the *k-simple best response utility*, given by

$$\bar{u}_i^k(v_i) := \sup_{b'_i \in \mathcal{B}^k} \bar{u}_i(v_i, b'_i). \tag{6}$$

A bid is a *k-simple best response* if it is in $\mathcal{B}^k$ and achieves the k-simple best response utility.[3] Analogously, let $\mathcal{B}^\infty$ denote the set of all complex bids. We call the highest possible expected utility that can be achieved with any bid in $\mathcal{B}^\infty$ the *complex best response utility* or simply the *best response utility*, given by

$$\bar{u}_i^\infty(v_i) := \sup_{b'_i \in \mathcal{B}^\infty} \bar{u}_i(v_i, b'_i). \tag{7}$$

A bid is a *complex best response* if it is in $\mathcal{B}^\infty$ and achieves the complex best response utility.

2.5 Bayes-Nash Equilibrium

For most of this paper, we consider the single-agent perspective, where one bidder finds a best response to the bid distribution $B_{\text{-}i}$, being agnostic about where this distribution comes from. One special setting (which we consider in Section 3) is the situation where several bidders have probabilistic knowledge of each others' values. Each bidder draws his valuation v_i from a value distribution V_i, and applies a *strategy* s_i to derive a bid $b_i := s_i(v_i)$. The strategy s_i is a function mapping valuations to bids. For bidders with complex bids, we can define the standard equilibrium concept for games with imperfect information:

Definition 2.5. A *Bayes-Nash equilibrium (BNE)* is a strategy profile $s = (s_1, \ldots, s_n)$ for which the following holds:

$$\forall i, \forall v_i : \bar{u}_i(v_i, s_i(v_i)) = \bar{u}_i^\infty(v_i), \tag{8}$$

where the bid distribution $B_{\text{-}i}$ (implicitly used in $\bar{u}_i$ and $\bar{u}_i^\infty$) is defined as $B_{\text{-}i} = s_{\text{-}i}(V_{\text{-}i})$.

For bidders who do not necessarily play complex bids, but who only play k-simple bids, we can define a corresponding equilibrium concept.

Definition 2.6. A *k-simple BNE* is a strategy profile s with $s_i \in \mathcal{B}^k$ for which it holds that

$$\forall i, \forall v_i : \bar{u}_i(v_i, s_i(v_i)) = \bar{u}_i^k(v_i). \tag{9}$$

[3] We take the supremum over bids instead of the maximum, because the maximum might not exist due to discontinuities in the utility function. In that case, a best response is technically the limit of a series of bids.

3 AN EXAMPLE: UTILITY LOSS DUE TO SIMPLE BIDDING

We now develop a detailed example that showcases how a bidder might want to go beyond simple bidding and use a complex strategy that includes bids on items the bidder is not even interested in. The uncertainty that bidders have about each others' bids will play a crucial role, an issue which we explore further in Section 5.

Consider the following auction, which is a version of an LLG setting [Ausubel and Milgrom, 2006] with discrete value distributions. There are two local bidders and one global bidder, who are bidding on two items A and B. The global bidder has value 8 for the bundle AB. The local bidders are interested in the single item A or B, respectively, and have a value for their item of either 0 (low) or 10 (high), independently and uniformly at random. This gives us four possible realizations of the valuations: high-high, high-low, low-high and low-low.

The payment rule for our example will be VCG-nearest. For ease of exposition, if there are multiple efficient allocations, we break ties by maximizing the number of local bidders that win at least one item.

It is known that the global bidder has a dominant strategy to bid his true value on the bundle AB [Beck and Ott, 2013]. Furthermore, when one of the other bidders has a value of 0, quasi-linear utilities make it impossible for him to win with utility strictly higher than 0, so it is also a dominant strategy for him to bid truthfully. By definition of the core constraints, whenever the two local bidders win, there is a single constraint that forces the sum of their payments to equal 8. This makes these two bidders face an interesting coordination problem.

Consider the scenario where bidders are only allowed to make simple bids. After some inspection, it becomes clear that the strategies given in the following table form a simple BNE:

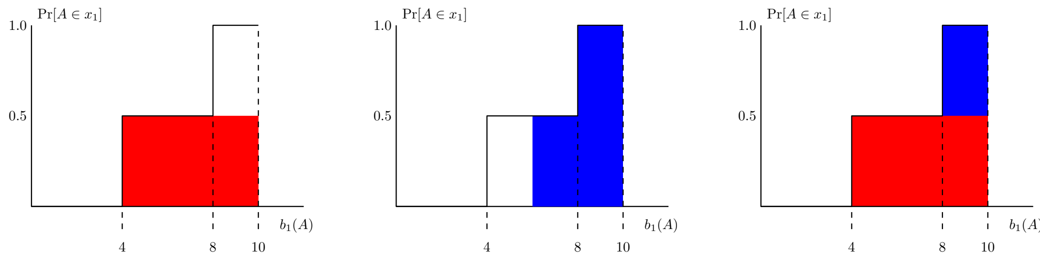
Bidders	Bids	Expected Utility
1 $(v_1 = 0)$	$\emptyset$	0
1 $(v_1 = 10)$	$\{(A, 4)\}$	$\frac{1}{2} \cdot (10 - 4) = 3$
2 $(v_2 = 0)$	$\emptyset$	0
2 $(v_2 = 10)$	$\{(B, 4)\}$	$\frac{1}{2} \cdot (10 - 4) = 3$
3	$\{(AB, 8)\}$	0

As a consequence of these equilibrium bids, the local bidders split the payment evenly between them in the high-high case, but do not win in the high-low and low-high cases, respectively. If bidder 1 increased his bid to 8 or higher, then bidder 2 could respond by lowering his bid to 0 and free-ride off of bidder 1's bid. This seems suboptimal: the local bidders had to give up a part of their utility (the case where each of them might have won separately) to achieve a stable split of the profits when they win together.

The situation becomes more interesting if we allow bidders to make complex bids with multiple atomic bids. In this case, each of the local bidders is incentivized to add an atomic bid on the bundle AB, resulting in the following strategies:

Bidders	Bids	Expected Utility
1 $(v_1 = 0)$	$\emptyset$	0
1 $(v_1 = 10)$	$\{(A, 4), (AB, 8)\}$	$\frac{1}{2} \cdot (10 - 8) + \frac{1}{2} \cdot (10 - 4) = 4$
2 $(v_2 = 0)$	$\emptyset$	0
2 $(v_2 = 10)$	$\{(B, 4), (AB, 8)\}$	$\frac{1}{2} \cdot (10 - 8) + \frac{1}{2} \cdot (10 - 4) = 4$
3	$\{(AB, 8)\}$	0

For a local bidder, this yields an expected utility of 4 in the high case: in the high-low case he wins alone with the bid on AB and pays 8, and in the high-high case he wins together with the other bidder and pays 4. This way, the bidders maintain the equal cost split already observed in the simple equilibrium. Even though each bidder bids high enough to be able to win alone, the



(a) Utility with simple bid $b_1 = \{(A, 4)\}$.

(b) Utility with simple bid $b_1 = \{(A, 8)\}$.

(c) Utility with complex bid $b_1 = \{(A, 4), (AB, 8)\}$.

Fig. 1. Visualization of the example auction from bidder 1's perspective. The step function plotted is the probability of winning item A as a function of the highest atomic bid that includes this item. The area under the curve is the maximal expected utility achievable, and is only fully captured with a complex bid.

payment split remains stable, because the high bids are made in a way that prevents free-riding. Note that in this complex BNE, the items are allocated efficiently, unlike in the simple BNE, where the local bidders' failure to coordinate causes the global bidder to win even when he does not have the highest value. In fact, no simple BNE is efficient in this auction (see Appendix A for a characterization of all simple BNEs).

In the next section, we will generalize the idea behind this example, and show very strong separation results between simple and complex bidding. For this, it will be helpful to discuss the underlying logic more deeply, and understand why the complex strategies shown above indeed form a BNE.

The general principle is as follows: in any auction, given bids $b_{\text{-}i}$, every bundle K has an associated VCG payment $\text{VCG}_i(b_{\text{-}i}, K)$ which bidder i cannot affect with his bid. The efficient allocation rule dictates that a bid can only become part of a winning allocation if the reported social welfare this bid adds is at least as large as that which it displaces. Therefore, $\text{VCG}_i(b_{\text{-}i}, K)$ is the minimum amount that bidder i must bid to win bundle K (assuming he makes no bid on any other bundle). Furthermore, for any core-selecting payment rule, bidder i's payment must be at least this winning threshold. It follows that i's highest possible utility with *any* strategic manipulation is simply

$$\max_K \; v_i(K) - \text{VCG}_i(K, b_{\text{-}i}), \tag{10}$$

the highest difference in utility between his value and the minimum payment for some bundle K (i.e., the *Vickrey Payoff* w.r.t. his true valuation and others' reported bids). In the complex BNE shown above, all bidders obtain this maximal payoff in every realization of $b_{\text{-}i}$; their utility is maximized *pointwise* and thus also *in expectation*. It follows that all bidders are playing optimal strategies, and thus we are in BNE.

Figure 1 shows our example graphically. The expected utility is a 2-dimensional area, resulting from the multiplication of a *payoff* with the *probability* that this payoff occurs. Bidder 1's bid is plotted on the x axis, and his probability of winning (relative to bidder 2's bid distribution) is plotted on the y axis. The area under the curve is the maximum expected utility that can be captured according to (10). The VCG payment rule (by definition) gives the entire area's worth of utility to a truthful bidder. In contrast, other payment rules might require the bidder to put in a lot of work to capture as much of this area as possible.

A low bid captures one portion of this area, and a high bid captures a different, partially overlapping portion.[4] When making a complex bid, bidder 1 can cover the entire area by submitting high and low atomic bids on two different bundles. For this to work, bidder 1's bid must be just high enough to win depending on whether bidder 2 submits a helping bid on bundle B. Since bidder 1's larger bid is associated to bundle AB which conflicts with bidder 2's bid on bundle B, the allocation rule is manipulated into dropping this bid from consideration when not desired. This pattern of bidding is equivalent to bidder 1 getting to place his bid after observing bidder 2's bid, which we might call "ex-post bidding".[5]

Note that with Ex-post bidding, bidders decrease their payment indirectly by manipulating the allocation rule into imposing a tight individual rationality constraint on the maximum payment. A different type of manipulation is *overbidding*, where bidders overreport their value for some bundles, to directly manipulate the payment rule into charging them less than it normally would [Beck and Ott, 2013, Bosshard et al., 2020]. This works for some minimum-revenue core selecting rules, where bidders' payments are linked to each other through core constraints: a bidder can drive up the payments of other bidders by expressing high interest in the items won by them, but not high enough to actually win those items himself. These two classes of manipulations are superficially similar because they both involve bids that are more complex than the bidders' true valuation. However, the underlying principle is fundamentally different. For instance, the latter type can never arise in first price auctions, where a bidder's payment only depends on his winning bid.

4 EXPONENTIAL SEPARATION OF SIMPLE AND COMPLEX BIDDING FOR SIMPLE VALUATIONS

In this section, we show the following surprising result: even when a bidder has a simple valuation, in the worst case, he can have a simple best response utility that is exponentially smaller than his complex best response utility. Our result applies to the two most commonly used payment rules, first-price and VCG-nearest.

Theorem 4.1. *There exists an auction family $F(m)$ with m items and using the First Price payment rule, such that for some i, v_i is 1-simple and*

$$\frac{\bar{u}_i^1(v_i)}{\bar{u}_i^\infty(v_i)} = \Theta\left(\frac{1}{2^m}\right). \tag{11}$$

Furthermore, this ratio cannot be more than $1/2^m$ for any auction family.

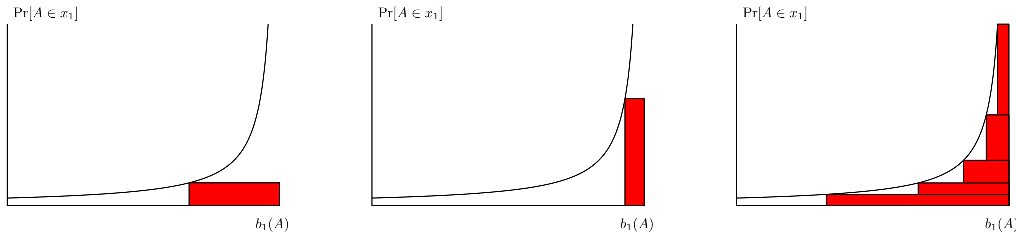
Theorem 4.2. *There exists an auction family $F'(m)$ with m items and using the VCG-nearest payment rule, such that for some i, v_i is 1-simple and*

$$\frac{\bar{u}_i^1(v_i)}{\bar{u}_i^\infty(v_i)} = \Theta\left(\frac{1}{2^m}\right). \tag{12}$$

The remainder of this section is structured as follows: In Section 4.1 we give an overview of the main ideas shared by both proofs. In Sections 4.2 and 4.3 we formally prove Theorems 4.1 and 4.2, respectively, and in Section 4.4 we discuss some implications.

[4] The area captured by the high bid is not rectangular, because VCG-nearest charges 4 and 6 in the high-high and high-low cases, respectively. In general, the area captured by a bid can have an arbitrarily complicated boundary on the left side. However, our constructions in Section 4 always make this area rectangular.

[5] A more computational way of thinking about this phenomenon is that bidder 1 does not bid a fixed numerical value, but (figuratively) injects a piece of code into the auction. This code gets "executed" by the allocation rule in an environment where information on bidder 2's bid is available.



(a) A low bid obtains a high payoff with low probability.

(b) A high bid obtains a low payoff with high probability.

(c) A complex bid obtains the highest possible payoff.

Fig. 2. Generalization of the example from Section 3. The area under the curve is shaped such that no simple bid can capture a large fraction of the total utility, but a complex bid with exponentially many atomic bids can capture the whole area.

4.1 Proof Ingredients for an Exponential Separation

How do we generalize the underlying idea of the example given in Section 3? We now give a brief overview of our technical approach.

First, instead of having just two different thresholds at which the probability of winning the desired item increases, we have a distribution over b_{-i} that generates exponentially many such thresholds. Packing one atomic bid for each such threshold into a single complex bid must be done carefully, so the atomic bids do not interfere with each other. We achieve this by associating each threshold with a different bundle of items. For a bundle K, the atomic bids on subsets of K are smaller than the bid on K itself, and thus dominated in the winner determination problem (to achieve this, the bids are ordered according to the lattice structure of the subset relation). Furthermore, when b_{-i} is drawn such that the threshold for bidder 1 is associated to bundle K, we ensure that all bundles that are not subsets of K overlap with items that are in very high demand, and the atomic bids on those bundles cannot win in the auction. This leaves the atomic bid on K as the only possible winner, and thus the bid amount can be chosen to match the winning threshold exactly.

To maximize the utility of the best response, we lay out these exponentially many thresholds along a geometric series, so the utility captured by each atomic bid only minimally overlaps with others. Conversely, to minimize the utility of the simple best response, we make the probability that bidder 1 wins his desired item inversely proportional to the payoff obtained by the respective bid, such that all simple bids yield the same (low) expected utility. See Figure 2 for a graphical representation of our construction.

The result of this is an exponential separation between the simple best response utility and the complex best response utility. The variant of the construction for VCG-nearest needs one more step: we add a few extra bidders to the auction to manipulate the rule into behaving badly for one of the bidders. This requires some additional technical work, e.g., a detailed analysis of all the core constraints that are generated.

Bidders		**Items**	
		Lattice Items	**Prize**
		$1 \dots m$	$m+1$
Protagonist	1		$f(i)$
Competitor	2	C	
Helper	3	$C - f(j)$ on $\sigma^{-1}(j)$	

Table 1. Bundle bids made by different bidder types in auction family $F(m+1)$. Bidder 1 bids works together with bidder 3 (the helper). If bidder 1 bids $f(i)$, they can jointly defeat the competitor whenever $i \geq j$. Bidder 1 can make additional bundle bids that include more items, and those bids are active whenever they do not overlap with the helpers' bid.

4.2 Proof of Theorem 4.1 (First-Price)

We now introduce an auction family, i.e., a single-parameter series of instances $F(m+1)$ of increasing size.[6] For this, note that any set of items together with the subset relation form a lattice (specifically the boolean lattice or hypercube).

Definition 4.3. Let $m \in \mathbb{N}$. For $i \in \mathbb{R}_+$, let $f(i) = 1 - \frac{1}{2^i}$, let C be a large constant, and let $\mathcal{L}$ be the lattice of $\{1, \dots, m\}$. Then, let σ be a bijective function such that

$$\sigma : \mathcal{L} \mapsto \{1, \dots, 2^m\} \tag{13}$$

$$K' \subset K \Rightarrow \sigma(K') > \sigma(K). \tag{14}$$

In other words, σ assigns a rank to each bundle that is a linearization of the partial order of $\mathcal{L}$. Then, the auction $F(m+1)$ is given as follows:

(1) The auction has 3 bidders and $m+1$ items.
(2) Bidder 1 has value 1 for bundle $\{m+1\}$.
(3) Bidder 2 always bids C on the bundle of all items.
(4) Bidder 3 bids $C - f(j)$ on bundle $\sigma^{-1}(j)$, where j is a random variable chosen from $\{1, \dots, 2^m\}$ with probability proportional to 2^j. This can be achieved by normalizing 2^j with the factor $\sum_{j=1}^{2^m} 2^j = 2^{2^m+1} - 2$.
(5) The allocation rule is efficient. For ease of presentation, ties are broken in favor of bidder 1 when relevant.
(6) The payment rule is first-price.

The bidders of this auction family and the relation of their bundles to each other are shown in Table 1. To understand the asymptotic behaviour of this auction family, we first need to understand the strategic landscape that bidder 1 is presented with. For this, we must formally define which bids even have a chance of winning.

Definition 4.4. Given a bid profile b, a bundle K is *active* for bidder 1 if $m+1 \in K$ and

$$W(b, N \setminus \{1\}, M \setminus K) = W(b, N \setminus \{1\}, M \setminus \{m+1\}). \tag{15}$$

In words, a bundle is active whenever it does not matter (for winner determination), whether bidder 1 makes an atomic bid on his original bundle of interest $\{m+1\}$ or on bundle K. In the latter

[6] We define the family in terms of the number of items $m+1$ (for $m \in \mathbb{N}$). This is because the smallest member of the family has 1 item, and furthermore this eases the notation. In Theorem 4.1 the offset of 1 can be folded into the constant factor implicit in the Landau notation Θ.

case, he would just win some unneeded items on top of item $m+1$. It follows that bidder 1 can combine any bid on an active bundle with the bid by bidder 3, to jointly overcome the threshold needed to win against bidder 2. Given this definition, we now characterize the auction's possible outcomes in the following lemma.

LEMMA 4.5. *In the auction family $F(m+1)$, let b_1 be a bid for bidder 1, and let K^* be the active bundle with the highest atomic bid in b_1. Furthermore, assume that bidder 1's bids on other bundles are not too high, i.e., $\forall K \neq K^* : b_1(K) \leq 1$.*

Then, bidder 1 wins bundle K^* if $b_1(K^*) \geq f(j)$. Otherwise, bidder 1's allocated bundle does not include item $m+1$.

PROOF. Because bidder 1's bids are capped at 1, he cannot win without the help of bidder 3. It is always a feasible allocation to give all goods to bidder 2, for a reported social welfare of C. This welfare is exceeded by a combination of bidder 1 and 3's bids exactly when the bid on the largest active bundle K^* is at least $f(i)$ for some $i \geq j$. □

Note that due to the first-price payment rule, bidder 1's payment is a function of only i, independent of the identity of K^*. Furthermore, note that if b_1 contains any atomic bid larger than 1, bidder 1 would either never win with this bid, making it irrelevant, or he would win with negative utility. Thus, we can assume WLOG that b_1 fulfills the assumption of Lemma 4.5. Having established the utility for any given realization of j, we can now move on to compute bidder 1's simple and complex best response utilities.

LEMMA 4.6. *In auction family $F(m+1)$, bidder 1's simple best response utility is*

$$\Theta\left(\frac{1}{2^{2^m}}\right). \tag{16}$$

PROOF. If bidder 1 can only bid on one bundle, then it is optimal for him to bid on bundle $\{m+1\}$ because bidding for fewer items wins more often and the payment when winning is independent of the bundle identity. Furthermore, his payment increases monotonically in i, so it is optimal to exactly bid on one of the bid levels $f(i)$ (for $i \in \mathbb{N}$) where the winning probability makes a discrete jump. Note that any bid between two bid levels (or above level $f(2^m)$) is strictly worse than the next lower bid level.

For a bid of $f(i)$, bidder 1 wins whenever $i \geq j$ and has a utility of $1 - (1 - \frac{1}{2^i}) = \frac{1}{2^i}$ (because the payment rule is first-price and utilities are quasi-linear). Recall that the probability distribution of j is proportional to 2^j. Thus, bidder 1's expected utility for any $1 \leq i \leq 2^m$ is

$$\sum_{j=1}^{i} \frac{2^j}{2^{2^m+1}-2} \cdot \frac{1}{2^i} \quad = \quad \frac{1}{2^{2^m+1}-2} \cdot \frac{1}{2^i} \sum_{j=1}^{i} 2^j \quad = \quad \Theta\left(\frac{1}{2^{2^m}}\right). \tag{17}$$

□

LEMMA 4.7. *In auction family $F(m+1)$, bidder 1's best response utility is*

$$\Theta\left(\frac{2^m}{2^{2^m}}\right). \tag{18}$$

PROOF. Since bidder 1's payment is independent of the winning bundle's identity, the utility of bidder 1 is maximized when he bids in such a way that for each j, he wins and pays the minimum amount possible for which he still wins. This bound can be achieved by bidding as follows: For each bundle K that contains item $m+1$ (i.e., the bundles *not* in the lattice $\mathcal{L}$), bidder 1 adds an atomic bid $(K, f(\sigma(M \setminus K)))$ to his bid.

This bid is just high enough to win when the signal is $j = \sigma(M \setminus K)$. Furthermore, this atomic bid does not win with any other realization of the signal.

We show this by case split. Case 1: We have that $j = \sigma(M \setminus K')$ for some $K' \not\supset K$. Since $K' \neq K$, there exists at least one item that is in K but not in K'. Any bundle bid on K cannot win because it is not active (it overlaps with bidder 3's bid on $M \setminus K'$) and would thus need to be at least C. Case 2: We have that $j = \sigma(M \setminus K')$ for some $K' \supset K$. The bundle bid on K is dominated (in the winner determination problem) by the bundle bid on K', because (due to σ respecting the partial order of $\mathcal{L}$ and f being monotonic), we have that

$$K' \supset K \Rightarrow M \setminus K' \subset M \setminus K \Rightarrow \sigma(M \setminus K') > \sigma(M \setminus K) \Rightarrow f(\sigma(M \setminus K')) > f(\sigma(M \setminus K)). \quad (19)$$

It follows that for each realization of j, the highest active bid is $f(j)$. Applying Lemma 4.5, the expected utility is thus

$$\sum_{j=1}^{2^m} \frac{2^j}{2^{2^m+1}-2} \cdot \frac{1}{2^j} \quad = \quad \frac{1}{2^{2^m+1}-2} \cdot \sum_{j=1}^{2^m} 1 \quad = \quad \Theta\left(\frac{2^m}{2^{2^m}}\right). \quad (20)$$

□

With these lemmas in place, we can now prove the exponential separation for the first price payment rule.

Proof of Theorem 4.1. Consider the family of auctions $F(m+1)$. The total number of items is $m+1 = \theta(m)$. Using Lemmas 4.6 and 4.7, the ratio of the simple and complex best response utilities of bidder 1 is

$$\Theta\left(\frac{1}{2^{2^m}} \middle/ \frac{2^m}{2^{2^m}}\right) = \Theta\left(\frac{1}{2^m}\right). \quad (21)$$

No ratio larger than 2^m is possible: this follows directly from the fact that payments under the first-price rule only depend on the winning bid. Thus, the overall expected utility of the best response cannot be higher than the number of bundle bids, 2^m, times the expected utility contributed by the single highest bundle bid. □

4.3 Proof of Theorem 4.2 (VCG-nearest)

The main difference between the proof of Theorems 4.1 and 4.2 is that we construct a more complicated auction family with more bidders, to make sure that the VCG-nearest payment behaves identically to first-price for bidder 1. This is achieved by moving the core far away from the VCG payments of other bidders, causing the projection onto the minimum revenue core to prioritize reducing the payments of those other bidders over that of bidder 1. Lemma 4.9 (the analogue of Lemma 4.5) must take into account many core constraints to characterize bidder 1's payments.

We adapt the family of auction instances $F(m+1)$ used in the previous section, as follows:

Definition 4.8. Let m, $\mathcal{L}$, σ, $f(i)$, C and j be given as in Definition 4.3. Then, the auction $F'(m+3)$ is given as follows:

(1) The auction has 5 bidders and $m+3$ items.
(2) Bidder 1 has value 1 for bundle $\{m+1\}$.
(3) Bidders 2 and 3 always bid C on bundles $\{m+2\}$ and $\{m+3\}$, respectively.
(4) Bidder 4 always bids C on bundle $\{m+1, m+2, m+3\}$.
(5) Bidder 5 bids C on bundle $\sigma^{-1}(j)$, and $C + f(j)$ on bundle $\sigma^{-1}(j) \cup \{m+1\}$.
(6) The allocation rule is efficient. For ease of presentation, ties are broken in favor of bidder 1 when relevant.
(7) The payment rule is VCG-nearest.

Bidders		Lattice Items $1 \ldots m$	Items Prize $m+1$	Helper Items $m+2$	$m+3$
Protagonist	1		$f(i)$		
Helper Bidders	2			C	
	3				C
	4		C	C	C
Competitor	5	C on $\sigma^{-1}(j)$			
		$C + f(j)$ on $\sigma^{-1}(j) \cup \{m+1\}$	$C + f(j)$ on $\sigma^{-1}(j) \cup \{m+1\}$		

Table 2. Bundle bids made by different bidder types in auction family $F'(m+3)$. Bidder 1 wins his desired item $m+1$ whenever $i \geq j$, i.e., he outbids the competitor. Bidder 1 can make additional bundle bids that include more items, and those bids are active whenever they do not overlap with $\sigma^{-1}(j)$.

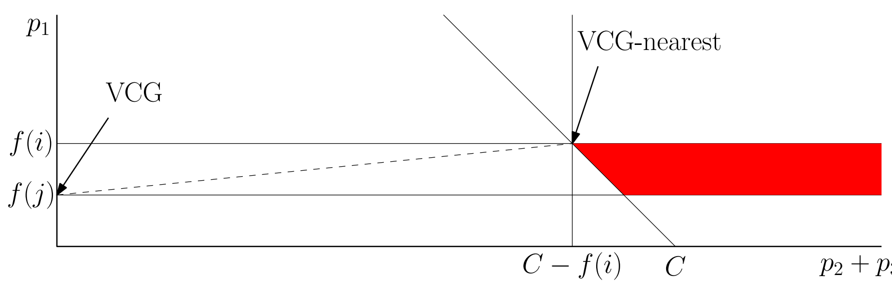


Fig. 3. The core in auction family $F'(m+3)$. Bidders 2 and 3 are symmetrical, so we show their joint payment on one axis. Since VCG-nearest minimizes L_2 distance to VCG, it is easy to see that the optimal solution is to charge bidder 1 as much as the individual rationality constraint will allow.

The bidders of this auction family and the relation of their bundles to each other are shown in Table 2. The first three bidders are in direct competition with bidder 4, and outbid him easily. However, bidder 1 must now also outbid bidder 5, for which he needs an atomic bid on an active bundle of at least $f(j)$.

The shape of the core is depicted in Figure 3, showing that bidder 1's payments are identical to first-price. The proof proceeds similarly to the first-price case.

Lemma 4.9. *In the auction family $F'(m+3)$, let b_1 be a bid for bidder 1, and let K^* be the bundle with the highest active bid in b_1. Furthermore, assume that bidder 1's bids on other bundles are not too high, i.e., $\forall K \neq K^* : b_1(K) \leq 1$.*

Then, bidder 1 wins bundle K^* if $b_1(K^*) \geq f(j)$, and his payment is

$$\min\left(b_1, \frac{C + 2f(j)}{3}\right). \tag{22}$$

Otherwise, bidder 1's allocated bundle does not include item $m+1$.

Proof. Given in Appendix B. □

For this particular auction, bidder 1's payment is a function of only j, independent of the identity of K^*, just like in the first-price case. Computing bidder 1's simple and complex best response utilities is now completely analogous to what we did in the previous section.

Lemma 4.10. *In auction family $F'(m+3)$, bidder 1's simple best response utility is*

$$\Theta\left(\frac{1}{2^{2^m}}\right).$$

Proof. Analogous to Lemma 4.6, substituting Lemma 4.5 with Lemma 4.9. □

Lemma 4.11. *In auction family $F'(m+3)$, bidder 1's best response utility is*

$$\Theta\left(\frac{2^m}{2^{2^m}}\right).$$

Proof. Analogous to Lemma 4.7, substituting Lemma 4.5 with Lemma 4.9. □

Proof of Theorem 4.2. Consider the family of auctions $F'(m+3)$. The total number of items is $m+3=\theta(m)$. Using Lemmas 4.10 and 4.11, the ratio of the simple and complex best response utilities of bidder 1 is

$$\Theta\left(\frac{1}{2^{2^m}}\middle/\frac{2^m}{2^{2^m}}\right)=\Theta\left(\frac{1}{2^m}\right). \tag{23}$$

□

4.4 Discussion

It is surprising that equally strong separations between simple and complex best response utility are possible for first-price and VCG-nearest. This can be traced back to our construction in Theorem 4.2, which spreads the possible payoff among winning bidders in a maximally unbalanced way, creating incentives identical to first-price for bidder 1.

One aspect where symmetry between the two payment rules breaks is in the tightness of the respective results. First-price payments are independent from losing bids, so it is straightforward to prove that our construction is the worst case scenario. The same is not clear for VCG-nearest, where it might be possible to construct an auction family where some atomic bids both capture part of the possible utility directly, and additionally reduce the bidder's payment when he wins with other bids.[7] Nevertheless, we believe that a more than exponential separation for VCG-nearest is unlikely, even though we cannot currently exclude this possibility.

It is possible that the full strategy space with 2^m atomic bids is not available to bidders in some auctions. For instance, spectrum auctions typically have a bound on the complexity of bids on the order of 500 atomic bids [Ausubel and Baranov, 2017]. Cognitive and computational limitations on the bidders' part can also arise. Our constructions are robust to such limitations: if a bidder is restricted to bids of complexity k, then he captures an amount of utility that is linear in k:

Proposition 4.12. *For the constructions in Theorems 4.1 and 4.2, we have that for any $k \leq 2^m$,*

$$\frac{\bar{u}_i^1(v_i)}{\bar{u}_i^k(v_i)}=\Theta\left(\frac{1}{k}\right). \tag{24}$$

This is clear for first-price where, due to independence from losing bids, the k-simple best response just consists of a subset of the complex best response. However, for VCG-nearest the k-simple best response could conceivably want to place losing bids designed to manipulate the payment rule into shifting some of the total payment from bidder 1 to bidders 2 and 3.

Luckily, such manipulations can be prevented by modifying the construction in an intuitive way: we add a small probability ε of bidders 2 and 3 dropping their bids to 0, and of bidder 4

[7] In fact, bids doing "double duty" like this have been observed in equilibrium with the theoretical bidder-i favored payment rule [Beck and Ott, 2013].

dropping his bid to $\frac{C}{2}$. For small enough ε, this does not change bidder 1's simple or complex best response. However, if bidder 1 makes any atomic bid above $\frac{C}{2}$ that includes items $m+2$ or $m+3$, his utility turns highly negative whenever all three of these events occur together, making any such manipulations unprofitable, and making the best response coincide with the one under first-price.

A further consideration is how robust our results are to our choice of bidding language. It is the case that Theorems 4.1 and 4.2 extend to the OR and OR* bidding languages [Nisan, 2006]. This is because all atomic bids of bidder 1 contain item $m+1$ and are thus mutually exclusive. For OR, some technical work is needed to reproduce the bid distribution $b_{\text{-}i}$, because other bidders are also restricted in their bidding expressiveness. The scaling behaviour between the simple and best responses expressed in Proposition 4.12 does not necessarily translate to these other bidding languages, though.

5 OPTIMALITY OF SIMPLE BIDDING WITHOUT UNCERTAINTY

Our results raise an important question: what is the main driver of the incentives for complex bidding that we have shown? The answer to this is the uncertainty that bidders have over others' bids. In the absence of such uncertainty and under reasonable payment rules, we can show that there is always a best response that is at most as complex as the true valuation.

By reasonable payment rule, we mean a rule that charges a payment somewhere between the VCG payment and the IR constraint. When the payment rule is within these design parameters, the optimal utility can always be achieved by placing a *profit target bid*:

Definition 5.1. Let v_i be a valuation, and b_i the simplest bid that fully declares v_i. For $\pi \in \mathbb{R}_{\geq 0}$, the *profit target bid* b_i^π is defined as

$$b_i^\pi := \{(K, \max(r - \pi, 0)) \mid (K, r) \in b_i\}. \tag{25}$$

Recall from Section 3 that $\text{VCG}_i(b_{\text{-}i}, K)$ is the minimum threshold required for i being able to win bundle K. Each such threshold is associated with a Vickrey payoff $v_i(K) - \text{VCG}_i(b_{\text{-}i}, K)$. To show that profit target bidding is optimal, we just need to show that bidder i can achieve the highest Vickrey payoff

$$\max_K v_i(K) - \text{VCG}_i(b_{\text{-}i}, K) \tag{26}$$

with a profit target bid. Due to our design constraints, other bids (no matter how complex) cannot exceed this utility. This insight leads to the following theorem:

THEOREM 5.2. *Let v_i be a k-simple valuation. Then, in any combinatorial auction setting where the bid distribution $B_{\text{-}i}$ is deterministic (i.e., there exists a $b_{\text{-}i}$ that occurs with probability 1) and the payment rule charges at least VCG, we have that*

$$\bar{u}_i^k(v_i) = \bar{u}_i^\infty(v_i). \tag{27}$$

PROOF. It is clear that any profit target bid b_i^π associated to a k-simple valuation v_i is also k-simple. Let K be one of the bundles that maximizes $v_i(K) - \text{VCG}_i(b_{\text{-}i}, K)$, and let $\pi := v_i(K) - \text{VCG}_i(b_{\text{-}i}, K) - \varepsilon$ for any $\varepsilon > 0$.

Under any allocation x with $b_i^\pi(x_i) > 0$, bidder i obtains at least π utility by IR and quasi-linearity. Furthermore, an allocation x with $b_i^\pi(x_i) = 0$ cannot be efficient, which we show by contradiction. If such an x was efficient, we would have that

$$W(b, N \setminus \{i\}, M) = W(b, N, M), \tag{28}$$

because i could be switched to winning the empty bundle without decreasing reported social welfare, and could thus be dropped entirely from the welfare calculation. We have from the definition of bundle K that $b_i^\pi(K) > \mathrm{VCG}_i(b_{\text{-}i}, K)$, which is equivalent to

$$W(b, N \setminus \{i\}, M) - W(b, N \setminus \{i\}, M \setminus K) < b_i^\pi(K), \tag{29}$$

It follows that

$$W(b, N \setminus \{i\}, M) < b_i^\pi(K) + W(b, N \setminus \{i\}, M \setminus K) \leq W(b, N, M), \tag{30}$$

where the last inequality holds because it is always a feasible allocation to give bundle K to i and allocate the rest of the items according to some $x \in \mathcal{F}_{M \setminus K}$ that maximizes the reported social welfare $W(b, N \setminus \{i\}, M \setminus K)$. Note that (28) contradicts (30), so b_i^π always obtains at least π utility. From this we can conclude that the k-simple utility reaches the Vickrey payoff in the limit $\varepsilon \longrightarrow 0$, i.e.,

$$\forall \varepsilon > 0 : \bar{u}_i^k(v_i) \geq v_i(K) - \mathrm{VCG}_i(b_{\text{-}i}, K) - \varepsilon. \tag{31}$$

Since the payment rule charges at least VCG, we also have that

$$\bar{u}_i^\infty(v_i) \leq v_i(K) - \mathrm{VCG}_i(b_{\text{-}i}, K), \tag{32}$$

and the claim follows from elementary analysis. □

6 CONCLUSION

In this paper, we have studied the complexity of bidding in one-shot CA mechanisms. We have focused on one specific aspect of "complexity": on how many bundles does a bidder have to submit a value report to maximize his utility. For the payment rules most commonly used in practice, first-price and VCG-nearest, we have shown that the large strategy spaces available to a bidder in a fully general CA allow him to manipulate the mechanism by submitting a bid that represents a highly complex valuation, even if the true valuation of the bidder is very simple. These manipulations can lead to an exponential increase in utility.

For us, it was particularly surprising that a minimum-revenue core-selecting payment rule does not protect from this manipulation at all, as demonstrated by our results which force VCG-nearest to behave identically to first-price for one of the bidders. Our construction extends trivially to many similar "reference point" payment rules [Bünz et al., 2018], e.g., to VCG-nearest using any L-norm with $L \geq 1$, and to 0-nearest using any L-norm with $L > 1$. It is an open question if there exists any minimum-revenue core selecting payment rule that systematically prevents these manipulations from being profitable. We conjecture that the nearest-bid payment rule [Ausubel and Baranov, 2019] significantly limits the potential gain from such manipulations, unless the core constraints prevent *any* minimum revenue core-selecting payment rule from doing so.

Our construction is an extreme example that we do not expect to encounter as-is in practice. However, the logical pathway underpinning it is quite robust, and we expect opportunities for more modest manipulations of similar type to naturally arise in auctions, specially in settings with highly heterogeneous bidders and relatively unstructured valuations. Simulations can be used to test such hypotheses, and this is a promising avuenue of future work, even though the difficulty of computing best responses in highly-dimensional spaces presents a significant obstacle at present.

More generally, our findings raise many questions in practical auction design, given that the strategic behaviour we describe can potentially interfere with efficiency, price discovery and legibility of the auction results, and might give a major advantage to sophisticated bidders. It is plausible that bidding languages could be designed to mitigate this issue, making it hard to report valuations that are unnatural or overly complex given the application domain, thus restoring the natural desideratum of simplicity.

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A BNE DERIVATION OF THE EXAMPLE IN SECTION 3

LEMMA A.1. *Let $r, r' \in \mathbb{R}$, and consider the strategy profile s^* given as*

$$s_1^*(v_1) = \begin{cases} \emptyset & v_1 = 0 \\ \{(A, r)\} & v_1 = 10 \end{cases} \tag{33}$$

$$s_2^*(v_2) = \begin{cases} \emptyset & v_2 = 0 \\ \{(B, r')\} & v_2 = 10 \end{cases} \tag{34}$$

$$s_3^*(v_3) = \{(AB, 8)\} \tag{35}$$

Then, s^* is a simple BNE of the auction given in Section 3 exactly when $r = r' = 4$, when $r = 0$ and $r' \geq 8$, and when $r' \geq 8$ and $r' = 0$.

PROOF. It is clear that truth-telling is a dominant action for $v_1 = 0$, for $v_2 = 0$ and for any v_3. Thus, we only need to show that $s_1^*(10)$ and $s_2^*(10)$ form simple best responses.

The case $r = 0$ is clearly a BNE: bidder 1 has expected utility

$$\bar{u}_i^1(10, 0) = \frac{1}{2} \cdot (10 - 0) = 5, \tag{36}$$

and he would need to bid at least 8 to win in the high-low case, reducing his utility to

$$\bar{u}_i^1(10, 8) = \frac{1}{2} \cdot (10 - 4) + \frac{1}{2} \cdot (10 - 8) = 4. \tag{37}$$

Bids between 0 and 8 or above 8 are weakly worse than these two bids. Bidder 2 prefers to keep bidding at least 8, because otherwise his expected utility drops to 0.

We next consider the cases where $0 < r \leq r'$. Here, $r + r' < 8$ cannot be a BNE, because it would be a better response for either of the local bidders to increase his bid to 8.

Conversely, $r + r' > 8$ also cannot be a BNE: If $r < 8$, it is a better response for bidder 1 to set $r = 8 - r'$. If $r \geq 8$, it is a better response for bidder 2 to set $r' = 0$.

It follows that $r + r' = 8$, and the expected utility of bidder 2 is

$$\frac{1}{2} \cdot (10 - r') = 5 - \frac{r'}{2} = 1 + \frac{r}{2}. \tag{38}$$

This bidder could deviate to $r' = 8$ obtaining utility

$$\frac{1}{2} \cdot (10 - 8) + \frac{1}{2} \cdot \left(10 - 8 + \frac{r}{2}\right) = 2 + \frac{r}{4} \tag{39}$$

because in the high-high case, the VCG payments would become $(0, 8-r)$, and thus the VCG-nearest payments would become $(\frac{r}{2}, 8 - \frac{r}{2})$.

This deviation is profitable when $r < 4$, so the only remaining possibility for a BNE is $r = r' = 4$, and it can be easily checked that this is indeed a BNE.

□

Note that there are some additional BNEs we have not enumerated in the above Lemma. First, there are some trivial modifications to the above BNEs: we can set $s_1(0) > 0$ and stay in BNE as long as the winning probability for bidder 1 remains 0. Furthermore in the asymmetric BNE where $r = 0$ and $r' = 8$, bidder 2 could equivalently bid on bundle AB instead of B. Second, there are other classes of BNEs where $s_3(v_3) > 8$. Such overbidding BNEs are also known to exist in single-item second-price auctions.

B PROOF OF LEMMA 4.9

Proof. Because bidder 1's bids are capped at 1, he can never win with a bid that is blocked by bidders 2,3 or 5. It is clear that bidders 2 and 3 always win their respective bundles, and bidder 4 never wins. It follows that item $m+1$ is won by one of the remaining two bidders. Bidder 1 outbids bidder 5 exactly when he has an active bundle on which he bids above $f(j)$.

Next, we derive the core constraints (and the VCG payments). There are 5 bidders and thus $2^5 - 1 = 31$ core constraints. Since bidder 4 always loses, any constraint corresponding to blocking coalition L with $4 \notin L$ is dominated by the constraint of coalition $L \cup \{4\}$. This accounts for 16 core constraints, leaving 15.

We next deal with the remaining constraints that involve bidder 5 on the left hand side. Let K' be the bundle with the highest bid of bidder 1 (not necessarily active). Then, the 8 constraints are:

$$p_5 \geq 2C + b_1(K') - 2C - b_1(K^*) = b_1(K') - b_1(K^*), \tag{40}$$
$$p_1 + p_5 \geq 2C - 2C = 0, \tag{41}$$
$$p_2 + p_5 \geq C + b_1(K') - C - b_1(K^*) = b_1(K') - b_1(K^*), \tag{42}$$
$$p_3 + p_5 \geq C + b_1(K') - C - b_1(K^*) = b_1(K') - b_1(K^*), \tag{43}$$
$$p_1 + p_2 + p_5 \geq C - C = 0, \tag{44}$$
$$p_1 + p_3 + p_5 \geq C - C = 0, \tag{45}$$
$$p_2 + p_3 + p_5 \geq C - b_1(K^*), \tag{46}$$
$$p_1 + p_2 + p_3 + p_5 \geq C - 0 = C. \tag{47}$$

Of these constraints, the first one is the VCG constraint, which dominates the 3rd and 4th constraints. The 7th and 8th constraints are dominated by constraints introduced below, and the rest are 0.

The remaining 7 core constraints are:

$$p_1 \geq 3C + f(j) - 3C = f(j), \tag{48}$$
$$p_2 \geq 2C + b_1(K^*) - 2C - b_1(K^*) = 0, \tag{49}$$
$$p_3 \geq 2C + b_1(K^*) - 2C - b_1(K^*) = 0, \tag{50}$$
$$p_1 + p_2 \geq 2C + f(j) - 2C = f(j), \tag{51}$$
$$p_1 + p_3 \geq 2C + f(j) - 2C = f(j), \tag{52}$$
$$p_2 + p_3 \geq 2C - C - b_1(K^*) = C - b_1(K^*), \tag{53}$$
$$p_1 + p_2 + p_3 \geq 2C - C = C. \tag{54}$$

The first of these is bidder 1's VCG constraint, which dominates the 4th and 5th.

In summary, the VCG payment vector is as follows:

$$\mathrm{VCG}_1 = f(j), \tag{55}$$
$$\mathrm{VCG}_2 = 0, \tag{56}$$
$$\mathrm{VCG}_3 = 0, \tag{57}$$
$$\mathrm{VCG}_4 = 0, \tag{58}$$
$$\mathrm{VCG}_5 = b_1(K') - b_1(K^*), \tag{59}$$

and the relevant core constraints (other than VCG and IR constraints) are:

$$p_2 + p_3 \geq C - b_1(K^*), \tag{60}$$
$$p_1 + p_2 + p_3 \geq C. \tag{61}$$

We now derive the VCG-nearest payments. It is clear that bidder 5's VCG-nearest payment is his VCG payment, since any increase in payment does nothing to help fulfill any of the other constraints. Furthermore, the VCG-nearest payment must be symmetric for bidders 2 and 3, since they only occur together in any constraint that is not dominated by another constraint. Any asymmetric payment vector in the core could be brought closer to VCG by balancing p_2 and p_3, without violating any constraints. Therefore, the VCG-nearest payment can be characterized by a single parameter Δ as follows:

$$p_1 = \Delta, \tag{62}$$

$$p_2 = \frac{C - \Delta}{2}, \tag{63}$$

$$p_3 = \frac{C - \Delta}{2}, \tag{64}$$

$$p_4 = 0, \tag{65}$$

$$p_5 = b_1(K') - b_1(K^*). \tag{66}$$

The distance from VCG to p in terms of Δ is

$$\|p - \mathrm{VCG}\|^2 = (\Delta - f(j))^2 + 2 \cdot \left(\frac{C - \Delta}{2}\right)^2. \tag{67}$$

Taking the derivative w.r.t. Δ, we get

$$\frac{d}{d\Delta} \|p - \mathrm{VCG}\|^2 = 2(\Delta - f(j)) - (C - \Delta) = 3\Delta - 2f(j) - C, \tag{68}$$

so the distance is minimized at

$$\Delta = \frac{2f(j) + C}{3}. \tag{69}$$

When $b_1(K^*)$ is below this threshold, this payment is outside the core (due to bidder 1's IR constraint). In that case, the derivative is negative over the whole domain $[0, b_1(K^*)]$, so the optimal solution is $\Delta = b_1(K^*)$.

□